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Dynamical decimation renormalization-group technique: Kinetic Gaussian model on nonbranching, branching, and multibranching Koch curves

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A generalizing formulation of dynamical real-space renormalization that is appropriate for arbitrary spin systems is suggested. The alternative version replaces single-spin flipping Glauber dynamics with single-spin transition dynamics. As an application, in this paper we mainly investigate the critical slowing down of the Gaussian spin model on three fractal lattices, including nonbranching, branching, and multibranching Koch curves. The dynamical critical exponent z is calculated for these lattices using an exact decimation renormalization transformation in the assumption of the magneticlike perturbation, and a universal result $z=1/\nu$ is found.

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I. INTRODUCTION

The dynamics of spin systems approaching their second-order phase transition points have been an important subject of many studies in the last few decades. One of the interesting phenomena is the critical slowing down characterized by a divergent relaxation time τ. A reasonable explanation is seemingly that the long range fluctuation leads to long time evolution of the order parameter. According to the dynamical scaling hypothesis [1], the divergent relaxation time τ and the divergent correlation length ζ can be related by $\tau\sim\zeta^{z}$, where z is the dynamical critical exponent and is believed to depend only on large universal features of the model Hamiltonian and the assumed dynamic process [2].

Obtaining the exact solution based on a master equation, except for a few cases [3–5], is not an easy job. One has to evaluate it by means of approximate methods, such as the Monte Carlo simulation, the high-temperature series expansion, etc. However, the success of renormalization-group (RG) methods [6,7] in obtaining the critical exponents and universality classes of static problems led to several attempts to use RG ideas in critical dynamics. One of the typical examples [2] is a generalization of the ε-expansion technique in which the dynamics is described by a Langevin equation. This method enables the calculation of the time-dependent correlation functions, but it is only useful near the upper critical dimension. Another [8] is a generalization of the real-space RG techniques, the starting point is a master equation instead of Hamiltonian. This method is preferable in discrete spin systems, and could be used to directly calculate the dynamical critical exponent. In addition, it is simple and transparent, and very accurate for certain systems so that it has been used quite extensively in the last years [9–15]. For other examples, see Refs. [16,17].

The dynamical real-space renormalization-group (DRSRG) technique, proposed by Achiam and Kosterlitz [8] and perfected by Kandel [18], is our focus in this paper. First, we establish a formulation of DRSRG applying to arbitrary spin systems. Then, we investigate the critical slowing down of the continuous spin model on different fractal lattices. In the generalizing formulation of DRSRG, we replace the single-spin flipping Glauber dynamics [3] with the single-spin transition dynamics [4], and use the same notation of Ref. [18] to express the critical dynamical exponent z.

During the last many years, scientific journals have published many papers concerning critical dynamics of discrete spin systems, but a systematic study of the critical dynamics in continuous spin systems is lacking indeed. This is the purpose of our latest papers [4,5] and this work is an attempt at filling this gap. This is just our main motivation. We realize the fact that, though the Gaussian model is certainly an idealization, it is interesting and simple enough to obtain some fundamental knowledge of dynamical process in cooperative systems. So this is an ideal dynamical model that interests us greatly. We also realize that, as an extension of the Ising model, the Gaussian model shows many differences from the Ising model in the properties of static phase transition, and yet its knowledge of the dynamical behavior is unclear. Within the framework of single-spin transition critical dynamics in our previous paper [4], we have obtained the dynamical critical exponent of the Gaussian model, $z=1/\nu=2$, at the critical point $K_c=b/2d$ based on rigorous calculations. This means that the dynamical exponent is highly universal on translational symmetric lattices. However, what is the dynamical exponent on dilational symmetric lattice systems? All of these subjects motivate us to finish this work.

This paper is organized as follows. Section II is a detailed description of the dynamical real-space renormalization group (DRSRG) technique in which the dynamics is described by a Markov process with the single-spin transition instead of the single-spin flipping. In Sec. III, the critical dynamics of the Gaussian model on three fractal lattices is studied. We take the exact decimation transformation and calculate the dynamical critical exponent z in the assumption

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of the magneticlike perturbation. Section IV is our summary and discussion.

II. DESCRIPTION OF THE METHOD

In the single-spin transition critical dynamics [4], the master equation can be written as

$$\frac{d}{dt}P(\{\sigma\},t)=-\sum_{i}\sum_{\hat{\sigma}}(1-\hat{p}_i)W_i(\sigma_i\rightarrow\hat{\sigma}_i)P(\{\sigma\},t), \tag{2.1}$$

where p_i is the transition operation defined by

$$p_i f(\sigma_1,\sigma_2,\ldots,\sigma_i,\ldots,\sigma_N,t)=f(\sigma_1,\sigma_2,\ldots,\hat{\sigma}_i,\ldots,\sigma_N,t),$$

and $W_i(\sigma_i\rightarrow\hat{\sigma}_i)$ is the single-spin transition probability that satisfies the following restraint conditions:

(a) ergodicity,

$$W_j(\sigma_j\rightarrow\hat{\sigma}_j)\neq 0,\quad \forall\ \sigma_j,\hat{\sigma}_j; \tag{2.2a}$$

(b) positivity,

$$W_j(\sigma_j\rightarrow\hat{\sigma}_j)\geqslant 0,\quad \forall\ \sigma_j,\hat{\sigma}_j; \tag{2.2b}$$

(c) normalization,

$$\sum_{\hat{\sigma}_j}W_j(\sigma_j\rightarrow\hat{\sigma}_j)=1,\quad \forall\ \sigma_j; \tag{2.2c}$$

(d) detailed balance,

$$\frac{W_j(\sigma_j\rightarrow\hat{\sigma}_j)}{W_j(\hat{\sigma}_j\rightarrow\sigma_j)}=\frac{P_{eq}(\sigma_1,\ldots,\hat{\sigma}_j,\ldots,\sigma_N)}{P_{eq}(\sigma_1,\ldots,\sigma_j,\ldots,\sigma_N)},\quad \forall\ \sigma_j,\hat{\sigma}_j, \tag{2.2d}$$

in which

$$P_{eq}(\{\sigma\})=\frac{1}{Z}\exp[-\beta\mathcal{H}(\{\sigma\})],$$

$$Z=\sum_{\{\sigma\}}\exp[-\beta\mathcal{H}(\{\sigma\})],$$

where P_{eq} is the equilibrium Boltzmann distribution function, Z the partition function and $\mathcal{H}(\{\sigma\})$ the system Hamiltonian. A well-chosen form of the transition probability is

$$W_i(\sigma_i\rightarrow\hat{\sigma}_i)=\frac{1}{Q_i}\exp\left[-\beta\mathcal{H}_i\left(\hat{\sigma}_i,\sum_{\langle i,j\rangle}\sigma_j\right)\right]=\frac{\exp\left[-\beta\mathcal{H}_i\left(\hat{\sigma}_i,\sum_{\langle i,j\rangle}\sigma_j\right)\right]}{\sum_{\hat{\sigma}_i}\exp\left[-\beta\mathcal{H}_i\left(\hat{\sigma}_i,\sum_{\langle i,j\rangle}\sigma_j\right)\right]}. \tag{2.3}$$

In order to study the critical slowing down, we can limit ourselves to the relaxation of an infinitely small perturbation from equilibrium. Following Achiam's idea [12], two selections can be considered, they are the magneticlike perturbation

$$P(\{\sigma\},t)=\left[1+\sum_i h_{q_i}(t)\sigma_i\right]P_{eq}(\{\sigma\}) \tag{2.4}$$

or the energylike perturbation

$$P(\{\sigma\},t)=\left[1+\sum_{\langle i,j\rangle}h^E_{q_i}(t)\sigma_i\sigma_j\right]P_{eq}(\{\sigma\}), \tag{2.5}$$

where q_i distinguishes between points that have different R (the order of ramification), $\langle i,j\rangle$ denotes a sum over nearest-neighbor pairs, and h_{q_i} and $h^E_{q_i}$ are the reduced external fields.

Based on these two considerations (2.4) or (2.5), the master equation (2.1) takes the following forms:

$$\frac{d}{dt}\sum_i h_{q_i}(t)\sigma_i P_{eq}(\{\sigma\})=-\sum_i\sum_{\hat{\sigma}_i}h_{q_i}(t)(\sigma_i-\hat{\sigma}_i)W_i(\sigma_i\rightarrow\hat{\sigma}_i)P_{eq}(\{\sigma\}), \tag{2.6}$$

or

$$\frac{d}{dt}\sum_{\langle i,j\rangle}h^E_{q_i}(t)\sigma_i\sigma_j P_{eq}(\{\sigma\})=-\sum_{\langle i,j\rangle}\sum_{\hat{\sigma}}h^E_{q_i}(t)(\sigma_i-\hat{\sigma}_i)\sigma_j W_i(\sigma_i\rightarrow\hat{\sigma}_i)P_{eq}(\{\sigma\}), \tag{2.7}$$

respectively. We can express further Eqs. (2.6) and (2.7) as the unitized matrix formulation

$$\frac{d}{dt}\mathbf{h}(t)\cdot\mathbf{\Lambda}(\sigma)P_{eq}(k,\{\sigma\})=-\mathbf{h}(t)\cdot\mathbf{\Omega}(k,\sigma)P_{eq}(k,\{\sigma\}), \tag{2.8}$$

where, $\mathbf{h}(t)$ is a row matrix, $\mathbf{\Lambda}(\sigma)$ and $\mathbf{\Omega}(k,\sigma)$ are column matrices.

The critical dynamical behavior of the system described by Eq. (2.8), can be studied using the dynamical real-space renormalization-group (DRSRG) technique. The DRSRG is composed of two stages. The first stage is the rescaling of the space by

$$x\rightarrow x'=Lx, \tag{2.9}$$

which is performed using a RG transformation (such as decimation or site-block transformation), where L is the length-rescaling factor. For example, in the case of the decimation transformation, the spins are divided into two groups $\{\sigma\}$ and $\{\mu\}$ under the control of a decimation operator $T(\mu,\sigma)$, then a trace over the $\{\sigma\}$ is performed. The process of decimation for a spin function $f(\{\sigma\})$ can be demonstrated as

$$R[f(\{\sigma\})]=\sum_{\{\sigma\}} T(\mu,\sigma)f(\{\sigma\})=f(\{\mu\}). \tag{2.10}$$

It is certain that we need to rescale the interaction parameter $k=J/k_\beta T$, and the spin μ, i.e.,

$$k\to k'=R(k)k, \quad \mu\to\mu'=\xi(k)\mu, \tag{2.11}$$

so as to keep on an invariant form of the probability distribution, $P'_{eq}(k',\{\mu'\})$, where $\xi(k)$ is the spin-rescaling factor.

With the decimation transformation (2.10), Eq. (2.8) takes the form

$$\frac{d}{dt}\mathbf{h}(t)\cdot\mathbf{R}_\Lambda(\mathbf{k})\cdot\mathbf{\Lambda}'(\mu',k')P'_{eq}(k',\{\mu'\}) = -\mathbf{h}(t)\cdot\mathbf{R}_\Omega(\mathbf{k})\cdot\mathbf{\Omega}'(\mu',k')P'_{eq}(k',\{\mu'\}), \tag{2.12}$$

where $\mathbf{\Lambda}'(\mu',k')$ and $\mathbf{\Omega}'(\mu',k')$ retain the original form. Taking the monomark

$$\mathbf{h}'(t,k')=\mathbf{h}(t)\cdot\mathbf{R}_\Lambda(\mathbf{k}),$$

which can be regarded as a RG transformation of the dynamic parameter $\mathbf{h}(t)$, Eq. (2.12) can be rewritten as

$$\frac{d}{dt}\mathbf{h}'(t,k')\cdot\mathbf{\Lambda}'(\mu',k')P'_{eq}(k',\{\mu'\}) = -\mathbf{h}'(t,k')\cdot[\mathbf{R}_\Lambda^{-1}(\mathbf{k})\cdot\mathbf{R}_\Omega(\mathbf{k})]\cdot\mathbf{\Omega}'(\mu',k')P'_{eq}(k',\{\mu'\}). \tag{2.13}$$

The second stage of the DRSRG is the rescaling of time by

$$t\to t'=L^{-z}t, \tag{2.14}$$

which should result in that Eq. (2.12) is restored to the invariant form of the master equation (2.8):

$$\frac{d}{dt'}\mathbf{h}'(t',k')\cdot\mathbf{\Lambda}'(\mu',k')P'_{eq}(k',\{\mu'\}) = -\mathbf{h}'(t',k')\cdot\mathbf{\Omega}'(\mu',k')P'_{eq}(k',\{\mu'\}). \tag{2.15}$$

We might encounter two different cases in carrying out Eq. (2.15). It can be realized via the following analyses.

First, for some homogeneous lattices with the same coordination number ($q_i=q$, $h_{q_i}=h$), $\mathbf{R}_\Lambda(k)$ and $\mathbf{R}_\Omega(k)$ are only 1×1 matrices. When the system approaches its critical point k_c, $\lambda\equiv\mathbf{R}_\Lambda(k_c)=\text{const}$, $\omega\equiv\mathbf{R}_\Omega(k_c)=\text{const}$, then from Eq. (2.13) we can see that the invariant form of the master equation (2.8) can be restored by preforming the time rescaling

$$t\to t'=L^{-z}t=\frac{t}{\lambda/\omega}, \tag{2.16}$$

and from here we can further obtain the dynamical critical exponent z

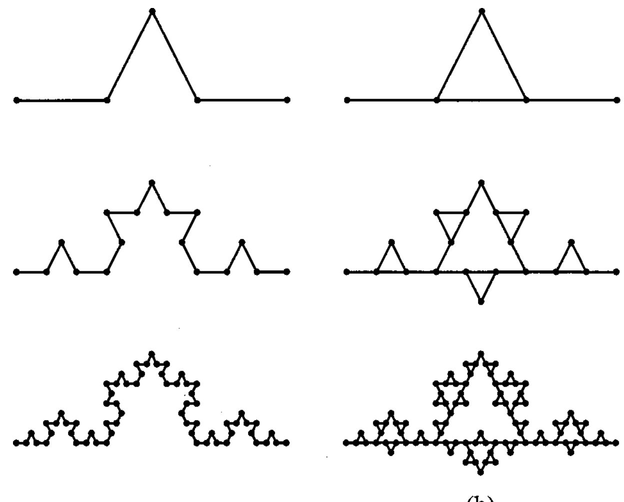

FIG. 1. Three stages in the construction of fractals. The generator of the fractal appears in the first line. The second and the third lines correspond to the second and thirds stages of iteration. Different fractals are placed in different columns. (a) A nonbranching Koch curve (NBKC) with $D_f=\ln 4/\ln 3$ and $R=2$. (b) A branching Koch curve (BKC) with $D_f=\ln 5/\ln 3$, $R_{\min}=2$ and $R_{\max}=3$.

$$z=\frac{\ln(\lambda/\omega)}{\ln L}. \tag{2.17}$$

Second, for some inhomogeneous lattices with different coordination number, $\mathbf{R}_\Lambda(k)$ and $\mathbf{R}_\Omega(k)$ are $m\times m$ square matrices in which the order m of the matrices depends on the number of the parameter h_{q_i}. In this case we have to look for the invariant form at the limit of the order $n\to\infty$ of the RG transformation. Because our starting point is very close to the fixed point k_c of the static RG transformation, the eigenvalues of the transformation matrices $\mathbf{R}_\Lambda(k\to k_c)$ and $\mathbf{R}_\Omega(k\to k_c)$ control the scaling properties as $t\to\infty$ [10,18]. Hence, (2.17) again determines the dynamic exponent, should λ/ω merely be replaced by $\lambda_{\max}/\omega_{\min}$ [18], i.e.,

$$z=\frac{\ln(\lambda_{\max}/\omega_{\min})}{\ln L}, \tag{2.18}$$

where $\lambda_{\max}$ is the largest eigenvalue of the matrix $\mathbf{R}_\Lambda(k)$, and $\omega_{\min}$ is the smallest eigenvalue of $\mathbf{R}_\Omega(k)$.

III. KINETIC GAUSSIAN MODEL ON THREE DIFFERENT FRACTAL GEOMETRIES

A. The Koch curve, the modified Gaussian model, and the master equation

The fractals [19] that we are going to study are constructed by an iterative procedure in which each segment of the object is replaced by a generator. Figure 1 shows two different configurations of the Koch curves [19] including the nonbranching and branching Koch curve. In the iteration, each stage of the iteration is described by a length-rescaling factor L, and the number of the segments in the lattice, N', increases to N by a relation $N/N'=L^{D_f}$, which defines the fractal dimensionality D_f. Obviously, these examples in Fig. 1 have different D_f, but their topological dimensionality D_T

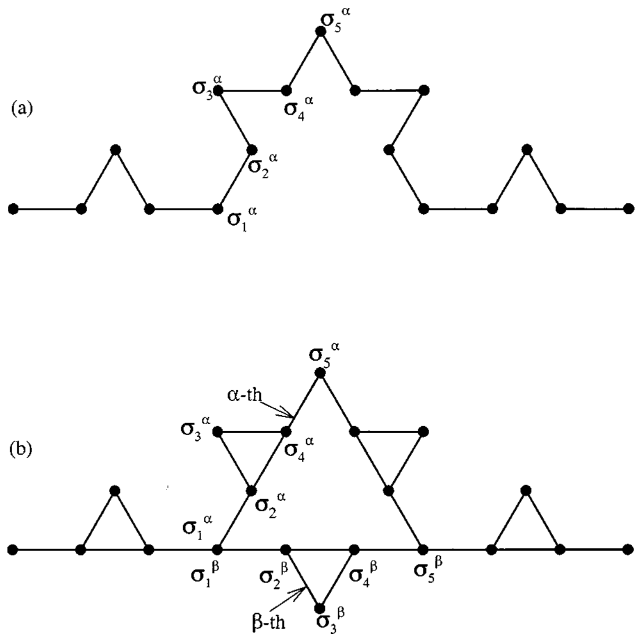

FIG. 2. The second construction stage of the NBKC and the BKC. (a) NBKC, all of the generations are the same; (b) BKC, there are two kinds of typical generators (such as αth and βth).

is equal to 1 which means that they are quasilinear fractals. Added to this, another parameter that is used to characterize the topological properties of the fractal is R, the order of ramification. The maximum and minimum values of R of a fractal obey the inequality, $R_{\max}\geqslant 2R_{\min}-2$ [20].

The examples of the $D_T=1$ fractals shown in Fig. 1 have finite R. The nonbranching Koch curve (NBKC), which has $D_f=\ln 4/\ln 3$ and $R_{\min}=R_{\max}=2$ shown in Fig. 1(a), is a homogeneous and wiggling chain, while the branching Koch curve (BKC), which has $D_f=\ln 5/\ln 3$ and $R_{\min}=2$ and $R_{\max}=3$ shown in Fig. 1(b), is an inhomogeneous one.

We assume that the Gaussian spin system with a reduced Hamiltonian

$$-\beta\mathcal{H}=k\sum_{\langle i,j\rangle}\sigma_i\sigma_j\,, \tag{3.1}$$

located on these fractals, where $\beta=1/k_\beta T$, $k=J/k_\beta T$, and the summation $\Sigma\langle i,j\rangle$ is taken over nearest neighbors. Unlike the Ising spin system, the spin of the Gaussian model can take any real value between $(-\infty,+\infty)$, and the Gaussian-type distribution finding a given spin between σ_i and $\sigma_i+d\sigma_i$

$$f(\sigma_i)d\sigma_i\sim\exp\left(-\frac{b_{q_i}}{2}\sigma_i^2\right)d\sigma_i \tag{3.2}$$

is assumed to prevent all spins from tending to infinity, where q_i is the coordination number of the site i, and b_{q_i} is a distribution constant independent of temperature. Considering the inhomogeneity of the branching Koch curve, we have assumed that the Gaussian-type distribution constants depend on coordination numbers and satisfy a certain relation

$$b_{q_i}/b_{q_j}=q_i/q_j\,. \tag{3.3}$$

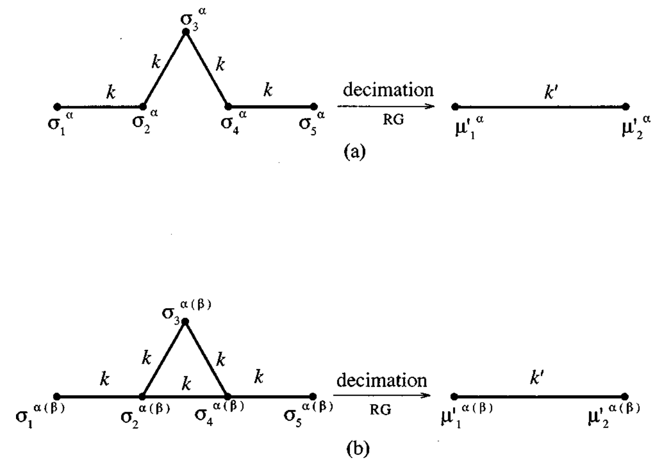

FIG. 3. Decimation RG procedure: (a) NBKC; (b) BKC.

This modified Gaussian model appeared in Ref. [21], which studied the static critical behavior of inhomogeneous fractal lattices.

In this case the spin transition probability can be expressed as

$$W_i(\sigma_i\to\hat{\sigma}_i)=\frac{1}{Q_i}\exp\left[k\hat{\sigma}_i\sum_w\sigma_{i+w}\right], \tag{3.4}$$

where the normalized factor Q_i can be determined as

$$\begin{aligned}Q_i&=\sum_{\hat{\sigma}_i}\exp\left[k\hat{\sigma}_i\sum_w\sigma_{i+w}\right]\\&=\int\exp\left[k\hat{\sigma}_i\sum_w\sigma_{i+w}\right]f(\hat{\sigma}_i)d\hat{\sigma}_i\\&=\exp\left[-\frac{k^2}{2b_{q_i}}\left(\sum_w\sigma_{i+w}\right)^2\right],\end{aligned}$$

and another useful combination formula can also be obtained

$$\begin{aligned}&\sum_{\hat{\sigma}_i}(\sigma_i-\hat{\sigma}_i)W_i(\sigma_i\to\hat{\sigma}_i)\\&\quad=\int_{-\infty}^{\infty}(\sigma_i-\hat{\sigma}_i)W_i(\sigma_i\to\hat{\sigma}_i)f(\hat{\sigma}_i)d\hat{\sigma}_i\\&\quad=\sigma_i-\frac{k}{b_{q_i}}\sum_w\sigma_{i+w}\,.\end{aligned} \tag{3.5}$$

So, for magneticlike perturbation, the master equation suitable for a modified Gaussian model on homogeneous and inhomogeneous fractal lattices can be written as

$$\frac{d}{dt}\sum_i h_{q_i}(t)\sigma_i P_{eq}(k,\{\sigma\}) = -\sum_i h_{q_i}(t)\left(\sigma_i - \frac{k}{b_{q_i}}\sum_w \sigma_{i+w}\right)P_{eq}(k,\{\sigma\}). \tag{3.6}$$

B. Nonbranching Koch curve

First let us focus on the homogeneous nonbranching Koch curve (NBKC) in which the Gaussian spins are placed on all of the sites. Because $h_{q_j}(t)=h(t)$, $b_{q_j}=b$, the master equation (3.6) takes the following form

$$\left(\frac{d}{dt}\right)h(t)\sum_\alpha\left(\frac{1}{2}\sigma_1^\alpha+\sigma_2^\alpha+\sigma_3^\alpha+\sigma_4^\alpha+\frac{1}{2}\sigma_5^\alpha\right)P_{eq}(k,\{\sigma\}) = -h(t)\left(1-\frac{2k}{b}\right)\sum_\alpha\left(\frac{1}{2}\sigma_1^\alpha+\sigma_2^\alpha+\sigma_3^\alpha+\sigma_4^\alpha + \frac{1}{2}\sigma_5^\alpha\right)P_{eq}(k,\{\sigma\}), \tag{3.7}$$

where α denotes generator of NBKC, which is shown in Fig. 2(a), the sum Σ_α goes over all generators, and $P_{eq}(k,\{\sigma\})$ is the equilibrium distribution function that can be written as

$$P_{eq}(k,\{\sigma\}) = \frac{1}{Z}\exp\left[k\sum_{\langle i,j\rangle}\sigma_i\sigma_j - \frac{b}{2}\sum_i \sigma_i^2\right] = \frac{1}{Z}\prod_\alpha \exp\left\{k(\sigma_1^\alpha\sigma_2^\alpha+\sigma_2^\alpha\sigma_3^\alpha+\sigma_3^\alpha\sigma_4^\alpha+\sigma_4^\alpha\sigma_5^\alpha) - \frac{b}{2}\left[\frac{1}{2}(\sigma_1^\alpha)^2+(\sigma_2^\alpha)^2+(\sigma_3^\alpha)^2+(\sigma_4^\alpha)^2 + \frac{1}{2}(\sigma_5^\alpha)^2\right]\right\}. \tag{3.8}$$

In Eqs. (3.7) and (3.8), the coefficient 1/2 comes from the fact that two neighboring generators share the same sites 1 and 5.

The space-rescaling procedure [see Fig. 3(a)] is performed through the decimation renormalization transformation

$$T^\alpha(\mu,\sigma) = \delta(\mu_1^\alpha - \sigma_1^\alpha)\,\delta(\mu_2^\alpha-\sigma_5^\alpha), \tag{3.9}$$

in which the spins σ_1^α, σ_5^α and the interaction k in the αth generator are replaced by rescaling spin $\mu_1'^\alpha$, μ_2^α and the interaction k', respectively, while the other spins σ_2^α, σ_3^α, and σ_4^α are integrated from $-\infty$ to $+\infty$. Under this process, the form of the distribution function P_{eq} is invariant. The details of the RG calculation are employed in Appendix A. Here, we give a renormalized master equation

$$\left(\frac{d}{dt}\right)\frac{1}{\xi(k)}\frac{b(b+2k)}{b^2-2k^2}h(t)\sum_\alpha\left(\frac{1}{2}\mu_1'^\alpha+\frac{1}{2}\mu_2'^\alpha\right)P'_{eq}(k',\{\mu'\}) = -\frac{1}{\xi(k)}\frac{b^2-4k^2}{b^2-2k^2}h(t)\sum_\alpha\left(\frac{1}{2}\mu_1'^\alpha+\frac{1}{2}\mu_2'^\alpha\right)P'_{eq}(k',\{\mu'\}), \tag{3.10}$$

where

$$\mu' = \xi(k)\mu = \left(\frac{b^4-4k^2b^2+2k^4}{b^2(b^2-2k^2)}\right)^{1/2}\mu, \tag{3.11}$$

$$k' = \frac{k^3 b}{b^4-4k^2b^2+2k^4}k. \tag{3.12}$$

Obviously, if the summation for α is arranged in the next stage of iteration, Eq. (3.10) can be rewritten as

$$\left(\frac{d}{dt}\right)\frac{1}{\xi(k)}\frac{b(b+2k)}{b^2-2k^2}h(t)\sum_\beta\left(\frac{1}{2}\mu_1'^\beta+\mu_2'^\beta+\mu_3'^\beta+\mu_4'^\beta+\frac{1}{2}\mu_5'^\beta\right)P'_{eq}(k',\{\mu'\}) = -\frac{1}{\xi(k)}\frac{b^2-4k^2}{b^2-2k^2}h(t)\sum_\beta\left(\frac{1}{2}\mu_1'^\beta+\mu_2'^\beta+\mu_3'^\beta+\mu_4'^\beta+\frac{1}{2}\mu_5'^\beta\right)P'_{eq}(k',\{\mu'\}). \tag{3.13}$$

Furthermore, if we let

$$\lambda = \frac{1}{\xi(k)}\frac{b(b+2k)}{b^2-2k^2}, \quad \omega = \frac{1}{\xi(k)}\frac{b^2-4k^2}{b^2-2k^2}\frac{1}{1-\dfrac{2k'}{b}}, \tag{3.14}$$

then by time rescaling

$$t' = \frac{t}{\lambda/\omega} = \frac{b^4-4k^2b^2+2k^4}{(b+2k)b^3}t = L^{-z}t \quad (L=3) \tag{3.15}$$

and the dynamic parameter transformation

$$h(t)\to h'(t') = \lambda h(t), \tag{3.16}$$

the invariant form of the master equation (3.8) can be restored

$$\left(\frac{d}{dt'}h'(t')\right)\sum_{\beta}\left(\frac{1}{2}\mu_1'^{\beta}+\mu_2'^{\beta}+\mu_3'^{\beta}+\mu_4'^{\beta}+\frac{1}{2}\mu_5'^{\beta}\right)P'_{eq}(k',\{\mu'\})$$
$$=-h'(t')\left(1-\frac{2k'}{b}\right)\sum_{\beta}\left(\frac{1}{2}\mu_1'^{\beta}+\mu_2'^{\beta}+\mu_3'^{\beta}+\mu_4'^{\beta}+\frac{1}{2}\mu_5'^{\beta}\right)P'_{eq}(k',\{\mu'\}). \tag{3.17}$$

Let the system be in its critical point $k_c=b/2$, which is determined by the recursion relationship (3.12), then we can obtain the dynamical critical exponent z by use of Eqs. (3.15) as

$$z=\left[\frac{1}{\ln L}\ln\frac{(b+2k)b^3}{b^4-4k^2b^2+2k^4}\right]_{k_c=b/2,L=3}=2\frac{\ln 4}{\ln 3}=2D_f. \tag{3.18}$$

However, because

$$\frac{1}{\nu}=\frac{1}{\ln L}\ln\left(\frac{dk'}{dk}\right)\bigg|_{k_c=b/2,L=3}$$
$$=\frac{1}{\ln L}\ln\left(\frac{-4k^3b^3(-b^2+2k^2)}{(b^4-4k^2b^2+2k^4)^2}\right)_{k_c=b/2,L=3}=\frac{\ln 16}{\ln 3}$$
$$=2\frac{\ln 4}{\ln 3}=2D_f, \tag{3.19}$$

then

$$z=\frac{1}{\nu}=2D_f=2.5237. \tag{3.20}$$

C. Branching Koch curve

Now, we turn to focus on the branching Koch curve (BKC), which is one of the inhomogeneous fractals. In this case, since the coordination number depends on the place of site, we must assume that the Gaussian-type distribution constants depend on coordination numbers and satisfy a certain relation (3.3), otherwise, the problem cannot be solved by applying the decimation RG method directly [21].

In the following we deal with the magneticlike perturbation master equation that suits the modified Gaussian model on inhomogeneous fractal lattices (3.6). We should notice that, for the branching Koch curve (BKC) there are two kinds of typical generators [such as αth and βth in Fig. 2(b)]: (1) $q_1=q_2=q_4=3,\ q_3=q_5=2$; (2) $q_1=q_2=q_4=q_5=3,\ q_3=2$. For case (1) or case (2), the decimation renormalization-group procedure is shown in Fig. 3(b), in which some spins such as σ_2, σ_3, and σ_4 are integrated, the remainders are rescaled as μ_1' and μ_2', and, at the same time, the interaction k is replaced by k'. Under this process, the form of the distribution function is invariant, and the RG transformation of αth generator is equivalent to βth. It can be realized via the calculation in Appendix B 1.

Our purpose is the renormalization of the master equation. In fact, we only need discuss a typical generator. Without loss of generality, we take the αth generator, for instance. The left and right sides of Eq. (3.6) can be written, respectively, as

$$\mathbf{h}(t)\cdot\mathbf{\Lambda}(\sigma)P_{eq}(k,\{\sigma\})=\left[h_3(t)\left(\frac{1}{3}\sigma_1^{\alpha}+\sigma_2^{\alpha}+\sigma_4^{\alpha}\right)+h_2(t)\left(\sigma_3^{\alpha}+\frac{1}{2}\sigma_5^{\alpha}\right)\right]P_{eq}(k,\{\sigma\}), \tag{3.21}$$

$$\mathbf{h}(t)\cdot\mathbf{\Omega}(k,\sigma)P_{eq}(k,\{\sigma\})=\left\{h_3\left[\frac{1}{3}\sigma_1^{\alpha}-\frac{k}{b_3}\sigma_2^{\alpha}\right]+h_3\left[\sigma_2^{\alpha}-\frac{k}{b_3}(\sigma_1^{\alpha}+\sigma_3^{\alpha}+\sigma_4^{\alpha})\right]+h_2\left[\sigma_3^{\alpha}-\frac{k}{b_2}(\sigma_2^{\alpha}+\sigma_4^{\alpha})\right]\right.$$
$$\left.+h_3\left[\sigma_4^{\alpha}-\frac{k}{b_3}(\sigma_2^{\alpha}+\sigma_3^{\alpha}+\sigma_5^{\alpha})\right]+h_2\left[\frac{1}{2}\sigma_5^{\alpha}-\frac{k}{b_2}\sigma_4^{\alpha}\right]\right\}P_{eq}(k,\{\sigma\}), \tag{3.22}$$

where, the coefficient 1/3 (or 1/2) in the terms σ_1^{α} (or σ_5^{α}) comes from the fact that three (or two) neighboring generators share the same site 1 (or 5).

Multiplying Eqs. (3.21) and (3.22) by the transformation operator

$$T(\mu,\sigma)=\prod_{\alpha}\ \delta(\mu_1^{\alpha}-\sigma_1^{\alpha})\,\delta(\mu_2^{\alpha}-\sigma_5^{\alpha}), \tag{3.23}$$

and integrating over $\{\sigma\}$, one can obtain (see Appendix B 2)

$$R\{\mathbf{h}(t)\cdot\mathbf{\Lambda}(\sigma)P_{eq}(k,\{\sigma\})\}=\frac{1}{\sqrt{\xi}}(h_3'(t)h_2'(t))\begin{pmatrix}\frac{1}{3}\,\mu_1'^{\alpha}\\ \frac{1}{2}\,\mu_2'^{\alpha}\end{pmatrix}P_{eq}'(\{k',\mu'\})=\mathbf{h}'(t,k')\cdot\mathbf{\Lambda}'(\mu',k')P_{eq}'(k',\{\mu'\}), \tag{3.24}$$

$$R\{\mathbf{h}(t)\cdot\mathbf{\Omega}(k,\sigma)P_{eq}(k,\{\sigma\})\}=\frac{1}{\sqrt{\xi}}(h_3'(t)h_2'(t))\frac{(R_\Omega)}{(R_\Lambda)}\begin{pmatrix}\frac{1}{3}\,\mu_1'^{\alpha}-\dfrac{k'}{b_3}\mu_2'^{\alpha}\\ \frac{1}{2}\,\mu_2'^{\alpha}-\dfrac{k'}{b_2}\mu_1'^{\alpha}\end{pmatrix}P_{eq}'(\{k',\mu'\})$$
$$=\mathbf{h}'(t,k')[\mathbf{R_\Lambda}^{-1}(\mathbf{k})\cdot\mathbf{R_\Omega}(\mathbf{k})]\mathbf{\Omega}'(\mu',k')P_{eq}'(k',\{\mu'\}), \tag{3.25}$$

in which

$$(h_3'(t)h_2'(t))=(h_3(t)h_2(t))(R_\Lambda),$$

$$(R_\Lambda)_{k\to k_c}=\begin{pmatrix}\dfrac{b_3b_2+2kb_2-2k^2}{b_3b_2-kb_2-2k^2} & \dfrac{2kb_2}{b_3b_2-kb_2-2k^2}\\[2ex] \dfrac{3k^2}{b_3b_2-kb_2-2k^2} & \dfrac{b_2(b_3-k)}{b_3b_2-kb_2-2k^2}\end{pmatrix}_{k\to k_c}=\begin{pmatrix}4 & 2\\ \frac{3}{2} & 2\end{pmatrix}\ \text{with eigenvalues}\ \ 5,1, \tag{3.26}$$

$$(R_\Omega)_{k\to k_c}=\begin{pmatrix}\dfrac{9b_2^4-28k^2b_2^2-8k^3b_2+8k^4}{(9b_2^3-16k^2b_2-8k^3)b_2} & 0\\[2ex] 0 & \dfrac{9b_2^4-28k^2b_2^2-8k^3b_2+8k^4}{(9b_2^3-16k^2b_2-8k^3)b_2}\end{pmatrix}_{k\to k_c}=\begin{pmatrix}\frac{3}{8} & 0\\ 0 & \frac{3}{8}\end{pmatrix}\ \text{with eigenvalues}\ \frac{3}{8},\frac{3}{8}, \tag{3.27}$$

where $k_c=b_2/2=b_3/3$ is determined by the fixed-point equation $k^*=k'=k$ (see Appendix B 1).

In this case we have to look for the invariant form of the master equation

$$\frac{d}{dt}\mathbf{h}(t)\cdot\mathbf{\Lambda}(\sigma)P_{eq}(k,\{\sigma\})=-\mathbf{h}(t)\cdot\mathbf{\Omega}(k,\sigma)P_{eq}(k,\{\sigma\}) \tag{3.28}$$

at the limit of the order $n\to\infty$ of the RG transformation. Because our starting point is very close to the fixed point k_c of the static RG transformation, the eigenvalues of the transformation matrices $\mathbf{R_\Lambda}(k\to k_c)$ and $\mathbf{R_\Omega}(k\to k_c)$ control the scaling properties of the largest relaxation time. Hence, according to the foregoing discussion, the dynamical critical exponent z is obtained

$$z=\frac{\ln(\lambda_{\max}/\omega_{\min})}{\ln L}=\frac{\ln 40/3}{\ln 3}=2.3578. \tag{3.29}$$

Because of

$$\left(\frac{dk'}{dk}\right)_{k=k_c}=\frac{d}{dk}\left(\frac{4b_2k^3(k+b_2)}{8k^4-8k^3b_2-28b_2^2k^2+9b_2^4}\right)_{k=k_c}=\frac{40}{3},$$

$$\frac{1}{\nu}=\frac{1}{\ln L}\ln\left(\frac{dk'}{dk}\right)_{k=k_c}=\frac{\ln 40/3}{\ln 3}, \tag{3.30}$$

then

$$z=\frac{1}{\nu}=2.3578. \tag{3.31}$$

D. Multibranching Koch curve

Based on the branching Koch curve, we now construct another generalized one, the multibranching Koch curve (MBKC), and investigate its critical dynamical behavior of the kinetic modified Gaussian model on this lattice. The constructional process is shown in Fig. 4. Obviously, it also is an inhomogeneous example of the $D_T=1$ fractals that have $D_f=\ln(2m+3)/\ln 3$, $R_{\min}=2$ and $R_{\max}=m+2$, $m=1,\ldots,\infty$.

The effect of the order of ramification, R, on the critical slowing down could be seen by this example.

Similarly, there are two kinds of typical generators in the multibranching Koch curve: (1) $q_1=q_3=q_4=1/(m+2)$, $q_i=2,(i=5,\ldots,m+4)$, $q_2=2$ (αth generator); (2) $q_1=q_3=q_4=q_2=1/(m+2)$, $q_i=2$, $(i=5,\ldots,m+4)$ (βth generator). In fact, the decimation renormalizing procedures of these two cases result in the same consequence (see Fig. 5). It can be realized via the calculation of Appendix C 1.

Our purpose is the renormalization of the master equation

$$\frac{d}{dt}\left(\sum_i h_{q_i}(t)\sigma_i\right)P_{eq}(k,\{\sigma\})=-\sum_i h_{q_i}(t)\left(\sigma_i-\frac{k}{b_{q_i}}\sum_w \sigma_{i+w}\right)P_{eq}(k,\{\sigma\}), \tag{3.32}$$

or

$$\frac{d}{dt}\mathbf{h}(t)\cdot\mathbf{\Lambda}(\sigma)P_{eq}(k,\{\sigma\})=-\mathbf{h}(t)\cdot\mathbf{\Omega}(k,\sigma)P_{eq}(k,\{\sigma\}). \tag{3.33}$$

In fact, we only need discuss a typical generator. Without loss of generality, we take case (1), for instance. The left and right sides of Eq. (3.32) can be written, respectively, as

$$\mathbf{h}(t)\cdot\mathbf{\Lambda}(\sigma)P_{eq}(k,\{\sigma\})=\left[h_{m+2}\left(\frac{1}{m+2}\sigma_1^\alpha+\sigma_3^\alpha+\sigma_4^\alpha\right)+h_2\left(\sum_{i=5}^{m+4}\sigma_i^\alpha+\frac{1}{2}\sigma_2^\alpha\right)\right]P_{eq}(k,\{\sigma\}) \tag{3.34}$$

and

$$\begin{aligned}\mathbf{h}(t)\cdot\mathbf{\Omega}(k,\sigma)P_{eq}(k,\{\sigma\})=&\left\{h_{m+2}\left[\frac{1}{m+2}\sigma_1^\alpha-\frac{k}{b_{m+2}}\sigma_3^\alpha\right]+h_{m+2}\left[\sigma_3^\alpha-\frac{k}{b_{m+2}}\left(\sigma_1^\alpha+\sum_{i=5}^{m+4}\sigma_i^\alpha+\sigma_4^\alpha\right)\right]+h_{m+2}\left[\sigma_4^\alpha-\frac{k}{b_{m+2}}\right.\right.\\&\left.\left.\times\left(\sigma_3^\alpha+\sum_{i=5}^{m+4}\sigma_i^\alpha+\sigma_2^\alpha\right)\right]+\sum_{i=5}^{m+4}h_2\left[\sigma_i^\alpha-\frac{k}{b_2}(\sigma_3^\alpha+\sigma_4^\alpha)\right]+h_2\left[\frac{1}{2}\sigma_2^\alpha-\frac{k}{b_2}\sigma_4^\alpha\right]\right\}P_{eq}(k,\{\sigma\}).\end{aligned} \tag{3.35}$$

By virtue of (C13)–(C17), the results of the decimation RG transformation of (3.34) and (3.35) are, respectively,

$$\begin{aligned}R\{\mathbf{h}(t)\cdot\mathbf{\Lambda}(\sigma)P_{eq}(k,\{\sigma\})\}&=\frac{1}{\sqrt{\xi}}(h'_{m+2}(t,k')h'_2(t,k'))\begin{pmatrix}\frac{1}{m+2}\mu_1'^{\alpha}\\ \frac{1}{2}\mu_2'^{\alpha}\end{pmatrix}P'_{eq}(k',\{\mu'\})\\&=\mathbf{h}'(t,k')\cdot\mathbf{\Lambda}'(\mu',k')P'_{eq}(k',\{\mu'\}),\end{aligned} \tag{3.36}$$

where

$$(h'_{m+2}h'_2)=(h_{m+2}\quad h_2)(R_\Lambda),$$

$$(R_\Lambda)=\begin{pmatrix}\frac{b_2b_{m+2}-b_2k-2mk^2+kb_2(m+2)}{(b_2b_{m+2}-b_2k-2mk^2)} & \frac{2b_2k}{b_2b_{m+2}-b_2k-2mk^2}\\ \frac{(m+2)mk^2}{b_2b_{m+2}-b_2k-2mk^2} & \frac{b_2b_{m+2}-b_2k}{b_2b_{m+2}-b_2k-2mk^2}\end{pmatrix}$$

and

$$\begin{aligned}R\{\mathbf{h}(t)\cdot\mathbf{\Omega}(k,\sigma)P_{eq}(k,\{\sigma\})\}&=\frac{1}{\sqrt{\xi}}(h'_{m+1}(t,k')h'_2(t,k'))\frac{(R_\Omega)}{(R_\Lambda)}\begin{pmatrix}\frac{1}{m+2}\mu_1'^{\alpha}-\frac{k'}{b_{m+2}}\mu_2'^{\alpha}\\ -\frac{k'}{b_2}\mu_1'^{\alpha}+\frac{1}{2}\mu_2'^{\alpha}\end{pmatrix}P'_{eq}(k',\{\mu'\})\\&=\mathbf{h}'(t,k')[\mathbf{R}_\Lambda^{-1}(\mathbf{k})\cdot\mathbf{R}_\Omega(\mathbf{k})]\cdot\mathbf{\Omega}'(\mu',k')P'_{eq}(k',\{\mu'\}),\end{aligned} \tag{3.37}$$

where

$$(R_\Omega)_{k=k_c}=\left[(\tilde{R}_\Omega)\begin{pmatrix}1 & -\frac{2k'}{b_{m+2}}\\ -\frac{(m+2)k'}{b_2} & 1\end{pmatrix}^{-1}\right],\quad (\tilde{R}_\Omega)=\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix},$$

$$a_{11}=\frac{-2b_{m+2}mk^3-b_{m+2}b_2k^2-2b_{m+2}^2mk^2+b_2b_{m+2}^3-b_2k^2(m+2)b_{m+2}+k^4(m+2)m}{(b_2b_{m+2}-b_2k-2mk^2)b_{m+2}(k+b_{m+2})},$$

$$a_{12}=\frac{2k^3(mk+b_2)}{b_{m+2}(k+b_{m+2})(2mk^2+b_2k-b_2b_{m+2})},$$

$$a_{21}=\frac{(m+2)k^3(mk+b_2)}{b_2(k+b_{m+2})(2mk^2+b_2k-b_2b_{m+2})},$$

$$a_{22}=\frac{2b_2mk^3+b_2^2k^2+2b_2b_{m+2}mk^2-b_2^2b_{m+2}^2+2b_{m+2}b_2k^2-2k^4m}{b_2(k+b_{m+2})(2mk^2+b_2k-b_2b_{m+2})},$$

$$(R_\Omega)_{k\to k_c}=\begin{pmatrix}\frac{m+2}{2(m+3)} & 0\\ 0 & \frac{m+2}{2(m+3)}\end{pmatrix}\ \text{with eigenvalues}\ \frac{m+2}{2(m+3)},\quad \frac{m+2}{2(m+3)},\tag{3.38}$$

$$(R_\Lambda)_{k\to k_c}=\begin{pmatrix}m+3 & 2\\ \frac{1}{2}m(m+2) & m+1\end{pmatrix}\ \text{with eigenvalues}\ 2m+3,1,\tag{3.39}$$

where k_c is determined by the fixed-point equation $k^*=k'=k$ [see (C12)] as

$$k_c=\frac{b_2}{2}=\frac{b_{m+2}}{m+2}.\tag{3.40}$$

From Eqs. (3.36) and (3.37) we can see that, in order to look for the invariant form of the master equation

$$\frac{d}{dt}\mathbf{h}(t)\cdot\mathbf{\Lambda}(\sigma)P_{eq}(k,\{\sigma\})=-\mathbf{h}(t)\cdot\mathbf{\Omega}(k,\sigma)P_{eq}(k,\{\sigma\}),\tag{3.41}$$

we need to do the renormalization endlessly up to the limit of the order $n\to\infty$ of the RG transformation. But, because the system is very close to the fixed point k_c of the static RG transformation, the scaling properties of the largest relaxation time are under the control of the largest eigenvalue $\lambda_{\max}=2m+3$ of the matrix $\mathbf{R}_\mathbf{\Lambda}(k\to k_c)$ and the smallest eigenvalue $\omega_{\min}=(m+2)/2(m+3)$ of $\mathbf{R}_\mathbf{\Omega}(k\to k_c)$. Hence, the invariant form of the master equation can be obtained through preforming the time rescaling

$$t\to t'=L^{-z}t=\frac{t}{\lambda_{\max}/\omega_{\min}},\tag{3.42}$$

and from here, the dynamical critical exponent z can be got

$$z=\frac{\ln(\lambda_{\max}/\omega_{\min})}{\ln L}=\frac{1}{\ln 3}\ln\frac{2(2m+3)(m+3)}{m+2}.\tag{3.43}$$

Because of

$$\left(\frac{dk'}{dk}\right)_{k\to k_c}=\frac{d}{dk}\left(\frac{4b_2k^3(b_2+mk)}{8k^4m-8b_2mk^3-4(m+2)b_2^2mk^2-4k^2b_2^2(m+2)-4k^2b_2^2+(m+2)^2b_2^4}\right)_{k\to b_2/2}$$

$$=2\frac{(m+3)(2m+3)}{m+2}$$

$$\frac{1}{\nu}=\frac{\ln\left(\frac{dk'}{dk}\right)_{k\to k_c}}{\ln L}=\frac{1}{\ln 3}\ln\frac{2(m+3)(2m+3)}{m+2}, \quad (3.44)$$

then

$$z=\frac{1}{\nu}=\frac{1}{\ln 3}\ln\frac{2(m+3)(2m+3)}{m+2}. \quad (3.45)$$

IV. CONCLUSIONS

Based on the dynamical real-space renormalization proposed by Achiam and Kosterlitz, we have suggested a generalizing formulation that suits arbitrary spin systems. The new version replaces the single-spin flipping Glauber dynamics with the single-spin transition dynamics. As an application, we focused on the kinetic Gaussian model ($\sigma=-\infty,\ldots,\infty$, continuous spin model), and studied three different fractal geometries with quasilinear lattices, including the nonbranching, branching, and multibranching Koch curve. We calculated the dynamical critical exponent z for these lattices using an exact decimation renormalization transformation in the assumption of the magneticlike perturbation, and found that it can be written universally as $z=1/\nu$, where ν is the static length-correlation exponent.

In the first example, the nonbranching Koch curve, $z=1/\nu=2D_f$, $D_f=\ln 4/\ln 3$ is the fractal dimensionality of the NBKC. Being a quasilinear chain, the geometrical effect of the wiggliness of the nonbranching Koch curve is that the correlation length $\tilde{\zeta}$ of the one-dimensional linear chain should be replaced by the real correlation length ζ, $\zeta=\tilde{\zeta}^{1/D_f}$ [12]. However, for a one-dimensional linear chain with Gaussian spin for each lattice, we have known that the critical dynamical exponent $\bar{z}=2$ [4]. So $\tau\sim\tilde{\zeta}^{\bar{z}}=(\zeta^{D_f})^{\bar{z}}=\zeta^z$ means $z=\bar{z}D_f=2D_f$. This result coincided with our calculation by DRSRG technique.

In the branching Koch curve, the result $z=1/\nu$ is also valid, but that $1/\nu=1/\ln L\ln(dk'/dk)_{k=k_c}=\ln 40/3/\ln 3$ is not simply related to the fractal dimensionality D_f.

In the multibranching Koch curve, the result $z=1/\nu$ is obtained once again. We can see from $z=1/\ln 3\ln[2(m+3)\times(2m+3)]/(m+2)$ that, when $m=0$, $z=2$, the lattice is a one-dimensional chain and z is equal to the rigorous result; when $m=1$, $z=\ln 40/3/\ln 3$, corresponding to the branching Koch curve; when $m\to\infty$, $z\to\infty$ (see Fig. 6). All of these mean that the critical slowing down of the Gaussian spin on the Koch curve is heavily dependent on the order of ramification R.

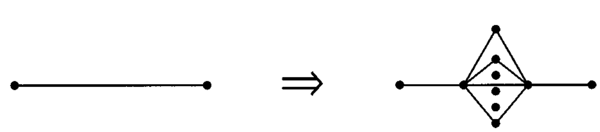

FIG. 4. The construction procedure of the multibranching Koch curve (MBKC) with $D_f=\ln(2m+3)/\ln 3$.

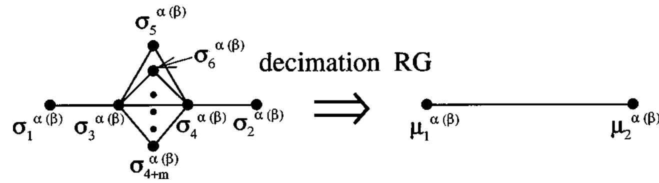

FIG. 5. Decimation RG procedure of the multibranching Koch curve (MBKC).

In fact, in our previous paper we have found that for a translational symmetric lattice with the Gaussian spin model, the critical dynamical exponent $z=1/\nu$, $\nu=1/2$ at the critical point $K_c=b/2d$ based on rigorous calculation [4]. Yet, in this paper the result $z=1/\nu$ has been proved once again by the dialational symmetric lattice systems. We guess that $z=1/\nu$, could be a universal conclusion for a kinetic Gaussian model. Of course, we must realize that the result of what we have obtained in this paper is carried out in the assumption of the magneticlike perturbation. However, the perturbation itself (magneticlike or energylike) is only a special assumption. For a general perturbation the master equation is not always invariant under the RG transformation because the perturbations probably have components along all the relevant operators. By this token, whether the $z=1/\nu$ will be a universal conclusion for a kinetic Gaussian model waits for further investigation.

ACKNOWLEDGMENTS

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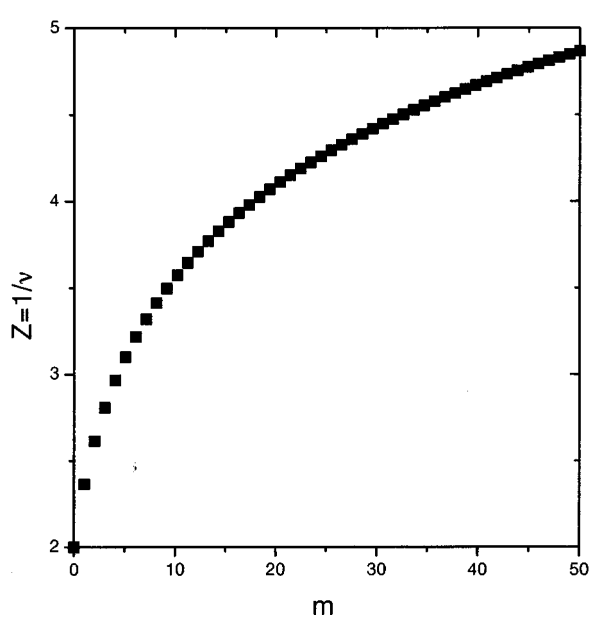

FIG. 6. Critical dynamical exponent of the kinetic Gaussian model on the multibranching Koch curve.

APPENDIX A: RG CALCULATION OF NBKC

To perform the decimation transformation, we have only to multiply both sides of the master equation (3.7) by the transformation operator

$$T(\mu,\sigma)=\prod_{\beta}\delta(\mu_1^{\beta}-\sigma_1^{\beta})\delta(\mu_2^{\beta}-\sigma_5^{\beta}) \tag{A1}$$

and integrate over $\{\sigma\}$, i.e.,

$$\left(\frac{d}{dt}\right)h(t)\sum_{\alpha}R\left\{\left(\frac{1}{2}\sigma_1^{\alpha}+\sigma_2^{\alpha}+\sigma_3^{\alpha}+\sigma_4^{\alpha}+\frac{1}{2}\sigma_5^{\alpha}\right)P_{eq}(k,\{\sigma\})\right\}$$
$$=-h(t)\left(1-\frac{2k}{b}\right)\sum_{\alpha}R\left\{\left(\frac{1}{2}\sigma_1^{\alpha}+\sigma_2^{\alpha}+\sigma_3^{\alpha}+\sigma_4^{\alpha}+\frac{1}{2}\sigma_5^{\alpha}\right)P_{eq}(k,\{\sigma\})\right\}, \tag{A2}$$

where

$$R\{\sigma_i^{\alpha}P_{eq}(k,\{\sigma\})\}=\int_{-\infty}^{\infty}d\sigma_1 d\sigma_2 \ldots d\sigma_N\prod_{\beta}\delta(\mu_1^{\beta}-\sigma_1^{\beta})\delta(\mu_2^{\beta}-\sigma_5^{\beta})\sigma_i^{\alpha}P_{eq}(k,\{\sigma\})$$
$$=R\{P_{eq}(k,\{\sigma\})\}\frac{W_{\sigma_i^{\alpha}}}{W}, \tag{A3}$$

$$W=\int_{-\infty}^{\infty}d\sigma_2^{\alpha}d\sigma_3^{\alpha}d\sigma_4^{\alpha}\exp\left\{k(\mu_1^{\alpha}\sigma_2^{\alpha}+\sigma_2^{\alpha}\sigma_3^{\alpha}+\sigma_3^{\alpha}\sigma_4^{\alpha}+\sigma_4^{\alpha}\mu_2^{\alpha})-\frac{b}{2}[(\sigma_2^{\alpha})^2+(\sigma_3^{\alpha})^2+(\sigma_4^{\alpha})^2]\right\}$$
$$=\sqrt{\frac{(2\pi)^3}{b(b^2-2k^2)}}\exp\left\{\frac{k^4}{b(b^2-2k^2)}\mu_1^{\alpha}\mu_2^{\alpha}+\frac{1}{2}\frac{k^2(b^2-k^2)}{b(b^2-2k^2)}[(\mu_1^{\alpha})^2+(\mu_2^{\alpha})^2]\right\},$$

$$W_{\sigma_i^{\alpha}}=\int_{-\infty}^{\infty}d\sigma_2^{\alpha}d\sigma_3^{\alpha}d\sigma_4^{\alpha}\sigma_i^{\alpha}\exp\left\{k(\mu_1^{\alpha}\sigma_2^{\alpha}+\sigma_2^{\alpha}\sigma_3^{\alpha}+\sigma_3^{\alpha}\sigma_4^{\alpha}+\sigma_4^{\alpha}\mu_2^{\alpha})-\frac{b}{2}[(\sigma_2^{\alpha})^2+(\sigma_3^{\alpha})^2+(\sigma_4^{\alpha})^2]\right\},$$

$$W_{\sigma_1^{\alpha}}=\mu_1^{\alpha}W,\quad W_{\sigma_5^{\alpha}}=\mu_2^{\alpha}W,$$

$$W_{\sigma_2^{\alpha}}=\frac{k}{b}\left[\mu_1^{\alpha}+\frac{k^2}{b^2-2k^2}(\mu_1^{\alpha}+\mu_2^{\alpha})\right]W,$$

$$W_{\sigma_3^{\alpha}}=\frac{k^2}{b^2-2k^2}(\mu_1^{\alpha}+\mu_2^{\alpha})W,$$

$$W_{\sigma_4^{\alpha}}=\frac{k}{b}\left[\mu_2^{\alpha}+\frac{k^2}{b^2-2k^2}(\mu_1^{\alpha}+\mu_2^{\alpha})\right]W.$$

The remanent integration $R\{P_{eq}(k,\{\sigma\})\}$ is an important one. We hope to keep on an invariant form of the transformational distribution function P'_{eq}

$$R\{P_{eq}(k,\{\sigma\})\}=\frac{1}{Z}\int_{-\infty}^{\infty}d\sigma_1 d\sigma_2 \ldots d\sigma_N\prod_{\beta}\delta(\mu_1^{\beta}-\sigma_1^{\beta})\delta(\mu_2^{\beta}-\sigma_5^{\beta})\exp\left[k\sum_{\langle i,j\rangle}\sigma_i\sigma_j-\frac{b}{2}\sum_i\sigma_i^2\right]$$
$$=\frac{1}{Z}\prod_{\beta}\int_{-\infty}^{\infty}d\sigma_2^{\beta}d\sigma_3^{\beta}d\sigma_4^{\beta}\exp\left\{k(\mu_1^{\beta}\sigma_2^{\beta}+\sigma_2^{\beta}\sigma_3^{\beta}+\sigma_3^{\beta}\sigma_4^{\beta}+\sigma_4^{\beta}\mu_2^{\beta})-\frac{b}{2}\left[\frac{1}{2}(\mu_1^{\beta})^2+(\sigma_2^{\beta})^2+(\sigma_3^{\beta})^2+(\sigma_4^{\beta})^2+\frac{1}{2}(\mu_2^{\beta})^2\right]\right\}=\frac{1}{Z}\prod_{\beta}\sqrt{\frac{(2\pi)^3}{b(b^2-2k^2)}}\exp\left\{\frac{k^4}{b(b^2-2k^2)}\mu_1^{\beta}\mu_2^{\beta}-\frac{b}{2}\frac{b^4-4k^2b^2+2k^4}{b^2(b^2-2k^2)}\left[\frac{1}{2}(\mu_1^{\beta})^2+\frac{1}{2}(\mu_2^{\beta})^2\right]\right\}.$$

Obviously, one must rescale the spins μ_1^{α}, μ_2^{α} and interaction k so as to keep the equilibrium distribution function to be invariant

$$\mu'=\xi(k)\mu=\left(\frac{b^4-4k^2b^2+2k^4}{b^2(b^2-2k^2)}\right)^{1/2}\mu, \tag{A4}$$

$$k'=\frac{k^3b}{b^4-4k^2b^2+2k^4}k, \tag{A5}$$

then

$$R\{P_{eq}(k,\{\sigma\})\}=\frac{1}{Z}\prod_{\beta}\sqrt{\frac{(2\pi)^3}{b(b^2-2k^2)}}\exp\left\{k'\mu_1'^{\beta}\mu_2'^{\beta}-\frac{b}{2}\left[\frac{1}{2}(\mu_1'^{\beta})^2+\frac{1}{2}(\mu_2'^{\beta})^2\right]\right\}$$

$$=\frac{1}{Z'}\exp\left\{k'\sum_{\beta}\mu_1'^{\beta}\mu_2'^{\beta}-\frac{b}{2}\sum_{\beta}\left[\frac{1}{2}(\mu_1'^{\beta})^2+\frac{1}{2}(\mu_2'^{\beta})^2\right]\right\}$$

$$=P_{eq}'(\{k',\mu'\}). \tag{A6}$$

Equation (A5) is reputed to be the recursion relation that enables one to determine the fixed point of the static RG transformation k_c. Upon that, we have

$$R\{\sigma_1^{\alpha}P_{eq}(\{\sigma\})\}=\mu_1^{\alpha}P_{eq}'(k',\{\mu'\}), \tag{A7}$$

$$R\{\sigma_5^{\alpha}P_{eq}(\{\sigma\})\}=\mu_2^{\alpha}P_{eq}'(k',\{\mu'\}), \tag{A8}$$

$$R\{\sigma_2^{\alpha}P_{eq}(k,\{\sigma\})\}=\frac{k}{b}\left[\mu_1^{\alpha}+\frac{k^2}{b^2-2k^2}(\mu_1^{\alpha}+\mu_2^{\alpha})\right]P_{eq}'(k',\{\mu'\}), \tag{A9}$$

$$R\{\sigma_3^{\alpha}P_{eq}(k,\{\sigma\})\}=\frac{k^2}{b^2-2k^2}(\mu_1^{\alpha}+\mu_2^{\alpha})P_{eq}'(k',\{\mu'\}), \tag{A10}$$

$$R\{\sigma_4^{\alpha}P_{eq}(k,\{\sigma\})\}=\frac{k}{b}\left[\mu_2^{\alpha}+\frac{k^2}{b^2-2k^2}(\mu_1^{\alpha}+\mu_2^{\alpha})\right]P_{eq}'(k',\{\mu'\}). \tag{A11}$$

Substituting Eqs. (A7)–(A11) into (A2), one can obtain

$$\left(\frac{d}{dt}\right)\frac{1}{\xi(k)}\frac{b(b+2k)}{b^2-2k^2}h(t)\sum_{\alpha}\left(\frac{1}{2}\mu_1'^{\alpha}+\frac{1}{2}\mu_2'^{\alpha}\right)P_{eq}'(k',\{\mu'\})=-\frac{1}{\xi(k)}\frac{b^2-4k^2}{b^2-2k^2}h(t)\sum_{\alpha}\left(\frac{1}{2}\mu_1'^{\alpha}+\frac{1}{2}\mu_2'^{\alpha}\right)P_{eq}'(k',\{\mu'\}). \tag{A12}$$

Obviously, if the summation for α is arranged in the next stage of iteration, Eq. (A12) can be written as

$$\left(\frac{d}{dt}\right)\frac{1}{\xi(k)}\frac{b(b+2k)}{b^2-2k^2}h(t)\sum_{\beta}\left(\frac{1}{2}\mu_1'^{\beta}+\mu_2'^{\beta}+\mu_3'^{\beta}+\mu_4'^{\beta}+\frac{1}{2}\mu_5'^{\beta}\right)P_{eq}'(k',\{\mu'\})$$

$$=-\frac{1}{\xi(k)}\frac{b^2-4k^2}{b^2-2k^2}h(t)\sum_{\beta}\left(\frac{1}{2}\mu_1'^{\beta}+\mu_2'^{\beta}+\mu_3'^{\beta}+\mu_4'^{\beta}+\frac{1}{2}\mu_5'^{\beta}\right)P_{eq}'(k',\{\mu'\}). \tag{A13}$$

It is just Eq. (3.13).

APPENDIX B: RG CALCULATION OF BKC

1. The RG transformation of the αth generator is equivalent to the βth generator

We can show that the RG transformation of the αth generator is equivalent to the βth, but the precondition is that the Gaussian-type distribution constants depend on the coordination number and satisfy the relation (3.3). It can be realized via the following calculations.

The effective Hamiltonian of the βth generator is

$$-\frac{1}{k_BT}\mathcal{H}^{\beta}_{eff}(\sigma,k)=k(\sigma_1^\beta\sigma_2^\beta+\sigma_2^\beta\sigma_3^\beta+\sigma_2^\beta\sigma_4^\beta+\sigma_3^\beta\sigma_4^\beta+\sigma_4^\beta\sigma_5^\beta)-\frac{b_3}{2}\left[\frac{1}{3}(\sigma_1^\beta)^2+(\sigma_2^\beta)^2+(\sigma_4^\beta)^2+\frac{1}{3}(\sigma_5^\beta)^2\right]-\frac{b_2}{2}(\sigma_3^\beta)^2 \tag{B1}$$

where, the coefficient 1/3 in the terms $(\sigma_1^\beta)^2$ and $(\sigma_5^\beta)^2$ comes from the fact that three neighboring generators share the same sites 1 and 5. We take the decimation renormalization transformation operator as

$$T^\beta(\mu,\sigma)=\delta(\mu_1^\beta-\sigma_1^\beta)\,\delta(\mu_2^\beta-\sigma_5^\beta), \tag{B2}$$

then, by integrating spins σ_2, σ_3, and σ_4 from $-\infty$ to $+\infty$, one obtains

$$R\left\{\exp\left[-\frac{1}{k_BT}\mathcal{H}^{\beta}_{eff}(\sigma,k)\right]\right\}=\int_{-\infty}^{\infty}d\sigma_1\,d\sigma_2\,\ldots\,d\sigma_5 T^\beta(\mu,\sigma)\exp\left[-\frac{1}{k_BT}\mathcal{H}^{\beta}_{eff}(\sigma,k)\right]$$

$$=C\exp\left\{k_0\mu_1^\beta\mu_2^\beta-\frac{b_3}{2}\xi\left[\frac{1}{3}(\mu_1^\beta)^2+\frac{1}{3}(\mu_2^\beta)^2\right]\right\}, \tag{B3}$$

where

$$C=\sqrt{\frac{(2\pi)^3}{b_2b_3^2(1-k/b_3-2k^2/b_2b_3)}},$$

$$k_0=\frac{k^3(k+b_2)}{(b_2b_3-kb_2-2k^2)(k+b_3)},$$

$$\xi=\frac{3k^4-2k^3b_3-4k^2b_2b_3-2b_3^2k^2+b_2b_3^3}{b_3(b_2b_3-kb_2-2k^2)(k+b_3)},$$

if we take

$$\mu'=\sqrt{\xi}\mu=\sqrt{\frac{3k^4-2k^3b_3-4k^2b_2b_3-2b_3^2k^2+b_2b_3^3}{b_3(b_2b_3-kb_2-2k^2)(k+b_3)}}\mu, \tag{B4}$$

$$k'=\frac{k_0}{\xi}=\frac{k^3(k+b_2)b_3}{3k^4-2b_3k^3-4k^2b_2b_3-2k^2b_3^2+b_2b_3^3}, \tag{B5}$$

then

$$R\left\{\exp\left[-\frac{1}{k_BT}\mathcal{H}^{\beta}_{eff}(\sigma,k)\right]\right\}=C\exp\left\{k'\mu_1'^\beta\mu_2'^\beta-\frac{b_3}{2}\left[\frac{1}{3}(\mu_1'^\beta)^2+\frac{1}{3}(\mu_2'^\beta)^2\right]\right\}. \tag{B6}$$

The same as above, the effective Hamiltonian of the αth generator is

$$-\frac{1}{k_BT}\mathcal{H}^{\alpha}_{eff}(\sigma,k)=k(\sigma_1^\alpha\sigma_2^\alpha+\sigma_2^\alpha\sigma_3^\alpha+\sigma_2^\alpha\sigma_4^\alpha+\sigma_3^\alpha\sigma_4^\alpha+\sigma_4^\alpha\sigma_5^\alpha)-\frac{b_3}{2}\left[\frac{1}{3}(\sigma_1^\alpha)^2+(\sigma_2^\alpha)^2+(\sigma_4^\alpha)^2\right]-\frac{b_2}{2}\left[(\sigma_3^\alpha)^2+\frac{1}{2}(\sigma_5^\alpha)^2\right], \tag{B7}$$

where, the coefficient 1/3 (or 1/2) in the terms $(\sigma_1^\alpha)^2$ [or $(\sigma_5^\alpha)^2$] comes from the fact that three (or two) neighboring generators share the same site 1 (or 5). By integrating spins σ_2, σ_3, and σ_4 from $-\infty$ to $+\infty$, one obtains

$$R\left\{\exp\left[-\frac{1}{k_BT}\mathcal{H}^{\alpha}_{eff}(\sigma,k)\right]\right\}=\int_{-\infty}^{\infty}d\sigma_1\,d\sigma_2\,\ldots d\sigma_5 T^{\alpha}(\mu,\sigma)\exp\left[-\frac{1}{k_BT}\mathcal{H}^{\alpha}_{eff}(\sigma,k)\right]$$
$$=C\exp\left\{k_0\mu_1^{\alpha}\mu_2^{\alpha}-\frac{b_3}{2}\xi_1\frac{1}{3}(\mu_1^{\alpha})^2-\frac{b_2}{2}\xi_2\frac{1}{2}(\mu_2^{\alpha})^2\right\}, \tag{B8}$$

where

$$k_0=\frac{k^3(k+b_2)}{(b_2b_3-kb_2-2k^2)(k+b_3)},$$

$$\xi_1=1-3k^2/b_3^2-\frac{3k^4/b_3^4}{1-k^2/b_3^2}-\frac{3k^4/b_2b_3^3}{(1-k/b_3)(1-k/b_3-2k^2/b_2b_3)},$$

$$\xi_2=1-\frac{2k^2/b_2b_3}{1-k^2/b_3^2}-\frac{2k^4/b_2^2b_3^2}{(1-k/b_3-2k^2/b_2b_3)(1-k/b_3)}.$$

Based on the relation (3.3), i.e.,

$$b_2/b_3=2/3,$$

we can see

$$\xi_1=\xi_2=\xi=\frac{8k^4-8k^3b_2-28b_2^2k^2+9b_2^4}{b_2(3b_2^2-2kb_2-4k^2)(2k+3b_2)}.$$

So, if only

$$\mu'=\sqrt{\xi}\mu=\sqrt{\frac{8k^4-8k^3b_2-28b_2^2k^2+9b_2^4}{b_2(3b_2^2-2kb_2-4k^2)(2k+3b_2)}}\mu, \tag{B9}$$

$$k'=\frac{k_0}{\xi}=\frac{4b_2k^3(k+b_2)}{8k^4-8k^3b_2-28b_2^2k^2+9b_2^4}, \tag{B10}$$

one can get

$$R\left\{-\frac{1}{k_BT}\mathcal{H}^{\alpha}_{eff}(\sigma,k)\right\}=C\exp\left\{k'\mu_1'^{\alpha}\mu_2'^{\alpha}-\frac{b_3}{2}\frac{1}{3}\times(\mu_1'^{\alpha})^2-\frac{b_2}{2}\frac{1}{2}(\mu_2'^{\alpha})^2\right\}. \tag{B11}$$

In fact, (B4)–(B5) coincides with (B9)–(B10). Solving the fixed-point equation $k^*=k'=k$, the critical point k_c is obtained

$$k_c=\frac{b_2}{2}=\frac{b_3}{3}.$$

Now, as the decimation renormalization transformation operator is taken as

$$T(\mu,\sigma)=\prod_{\alpha}\delta(\mu_1^{\alpha}-\sigma_1^{\alpha})\,\delta(\mu_2^{\alpha}-\sigma_5^{\alpha}),$$

from (B6) and (B11) we can see

$$R\{P_{eq}(k,\{\sigma\})\}=\frac{1}{Z'}\exp\left[k'\sum_{\langle i,j\rangle}\mu_i'\mu_j'-\frac{b_{q_i}}{2}\sum_i\mu_i'^2\right]$$
$$=P'_{eq}(k',\{\mu'\}). \tag{B12}$$

This means that the distribution function is invariant under RG transformation.

2. RG transformation of the master equation

Besides $R\{P_{eq}(k,\{\sigma\})\}$, one can also calculate

$$R\{\sigma_i^{\alpha}P_{eq}(k,\{\sigma\})\}=\sum_{\{\sigma\}}\{T(\mu,\sigma)\sigma_i^{\alpha}P_{eq}(k,\{\sigma\})\}$$
$$=P'_{eq}(k',\{\mu'\})\frac{W_{\sigma_i^{\alpha}}}{W} \tag{B13}$$

where

$$W=\int_{-\infty}^{\infty}d\sigma_2^{\alpha}d\sigma_3^{\alpha}d\sigma_4^{\alpha}\exp\left\{k(\mu_1^{\alpha}\sigma_2^{\alpha}+\sigma_2^{\alpha}\sigma_3^{\alpha}+\sigma_2^{\alpha}\sigma_4^{\alpha}+\sigma_3^{\alpha}\sigma_4^{\alpha}+\sigma_4^{\alpha}\mu_2^{\alpha})-\frac{b_3}{2}[(\sigma_2^{\alpha})^2+(\sigma_4^{\alpha})^2]-\frac{b_2}{2}(\sigma_3^{\alpha})^2\right\}$$

$$=\sqrt{\frac{(2\pi)^3}{b_3(b_2b_3-kb_2-2k^2)}}\exp\left\{\left(\frac{k^3(b_2+k)}{(b_2b_3-kb_2-2k^2)(b_3+k)}\mu_1^{\alpha}\mu_2^{\alpha}+\frac{k^2(b_2b_3-k^2)}{(b_2b_3-kb_2-2k^2)(b_3+k)}\times\left[\frac{1}{2}(\mu_1^{\alpha})^2+\frac{1}{2}(\mu_2^{\alpha})^2\right]\right)\right\}, \quad \text{(B14)}$$

$$W_{\sigma_i^{\alpha}}=\int_{-\infty}^{\infty}d\sigma_2^{\alpha}d\sigma_3^{\alpha}d\sigma_4^{\alpha}\sigma_i^{\alpha}\exp\left\{k(\mu_1^{\alpha}\sigma_2^{\alpha}+\sigma_2^{\alpha}\sigma_3^{\alpha}+\sigma_2^{\alpha}\sigma_4^{\alpha}+\sigma_3^{\alpha}\sigma_4^{\alpha}+\sigma_4^{\alpha}\mu_2^{\alpha})-\frac{b_3}{2}[(\sigma_2^{\alpha})^2+(\sigma_4^{\alpha})^2]-\frac{b_2}{2}(\sigma_3^{\alpha})^2\right\}. \quad \text{(B15)}$$

Upon that, we have

$$R\{\sigma_1^{\alpha}P_{eq}(\{k,\sigma\})\}=\frac{1}{\sqrt{\xi}}\mu_1'^{\alpha}P'_{eq}(\{k',\mu'\}), \quad \text{(B16)}$$

$$R\{\sigma_5^{\alpha}P_{eq}(\{k,\sigma\})\}=\frac{1}{\sqrt{\xi}}\mu_2'^{\alpha}P'_{eq}(\{k',\mu'\}), \quad \text{(B17)}$$

$$R\{\sigma_2^{\alpha}P_{eq}(\{k,\sigma\})\}=\frac{1}{\sqrt{\xi}}\frac{k(b_3b_2-k^2)\mu_1'^{\alpha}+k^2(k+b_2)\mu_2'^{\alpha}}{(k+b_3)(b_2b_3-kb_2-2k^2)}\times P'_{eq}(\{k',\mu'\}), \quad \text{(B18)}$$

$$R\{\sigma_3^{\alpha}P_{eq}(\{k,\sigma\})\}=\frac{1}{\sqrt{\xi}}\frac{k^2}{b_2b_3-kb_2-2k^2}\times(\mu_1'^{\alpha}+\mu_2'^{\alpha})P'_{eq}(\{k',\mu'\}), \quad \text{(B19)}$$

$$R\{\sigma_4^{\alpha}P_{eq}(\{k,\sigma\})\}=\frac{1}{\sqrt{\xi}}\frac{k^2(k+b_2)\mu_1'^{\alpha}+k(b_3b_2-k^2)\mu_2'^{\alpha}}{(k+b_3)(b_2b_3-kb_2-2k^2)}\times P'_{eq}(\{k',\mu'\}). \quad \text{(B20)}$$

By virtue of these integral results, Eqs. (3.24) and (3.25) can be obtained.

APPENDIX C: RG CALCULATION OF MBKC

1. The RG transformation of the αth generator is equivalent to the βth generator

For MBKC, we can also show that the RG transformation of the αth generator is equivalent to the βth, but the precondition is about the same as BKC, that the Gaussian-type distribution constants depend on the coordination number and satisfy the relation (3.3). It can be realized via the following calculations.

The effective Hamiltonian of the αth generator is

$$-\frac{1}{k_BT}\mathcal{H}_{eff}^{\alpha}(\sigma^{\alpha},k)=k\left[\sigma_1^{\alpha}\sigma_3^{\alpha}+\sigma_3^{\alpha}\sigma_4^{\alpha}+\sigma_4^{\alpha}\sigma_2^{\alpha}+\sum_{i=5}^{m+4}(\sigma_4^{\alpha}+\sigma_3^{\alpha})\sigma_i^{\alpha}\right]-\frac{b_{m+2}}{2}\times\left[\frac{1}{m+2}(\sigma_1^{\alpha})^2+(\sigma_3^{\alpha})^2+(\sigma_4^{\alpha})^2\right]-\frac{b_2}{2}\left[\sum_{i=5}^{m+4}(\sigma_i^{\alpha})^2+\frac{1}{2}(\sigma_2^{\alpha})^2\right], \quad \text{(C1)}$$

where, the coefficient $1/(m+2)$ (or 1/2) in the terms $(\sigma_1^{\alpha})^2$ [or $(\sigma_2^{\alpha})^2$] comes from the fact that $(m+2)$ (or 2) neighboring generators share the same site 1 (or 2). We take the decimation renormalization transformation operator as

$$T^{\alpha}(\mu,\sigma^{\alpha})=\delta(\mu_1^{\alpha}-\sigma_1^{\alpha})\delta(\mu_2^{\alpha}-\sigma_2^{\alpha}), \quad \text{(C2)}$$

then, by integrating spins σ_3, σ_4, and σ_i $(i=5,\ldots,m+4)$ from $-\infty$ to $+\infty$, one obtains

$$R\left\{\exp\left[-\frac{1}{k_BT}\mathcal{H}_{eff}^{\alpha}(\sigma^{\alpha},k)\right]\right\}=\int_{-\infty}^{\infty}\prod_{i=1}^{m+4}d\sigma_i^{\alpha}T^{\alpha}(\mu^{\alpha},\sigma^{\alpha})\exp\left[-\frac{1}{k_BT}\mathcal{H}_{eff}^{\alpha}(\sigma^{\alpha},k)\right]$$

$$=C\exp\left\{k_0\mu_1^{\alpha}\mu_2^{\alpha}-\frac{b_{m+2}}{2}\frac{1}{m+2}\xi_1(\mu_1^{\alpha})^2-\frac{b_2}{2}\frac{1}{2}\xi_2(\mu_2^{\alpha})^2\right\}, \quad \text{(C3)}$$

where

$$C=\left(\frac{2\pi}{b_2}\right)^{m/2}\sqrt{\frac{(2\pi)^2b_2}{(b_{m+2}+k)(b_2b_{m+2}-b_2k-2mk^2)}},$$

$$k_0=\frac{k^3(b_2+mk)}{(b_{m+2}+k)(b_2b_{m+2}-b_2k-2mk^2)},$$

$$\xi_1=\frac{m(m+2)k^4-2b_{m+2}mk^3-2b_{m+2}^2mk^2-b_2k^2(m+2)b_{m+2}-k^2b_2b_{m+2}+b_{m+2}^3b_2}{b_{m+2}(b_{m+2}+k)(b_2b_{m+2}-b_2k-2mk^2)},$$

$$\xi_2=\frac{-2k^2b_2b_{m+2}+2k^4m-2b_2b_{m+2}mk^2+b_{m+2}^2b_2^2-2b_2mk^3-k^2b_2^2}{b_2(b_{m+2}+k)(b_2b_{m+2}-b_2k-2mk^2)}.$$

Based on the relation (3.3), i.e.,

$$\frac{b_{m+2}}{b_2}=\frac{m+2}{2}, \tag{C4}$$

we can see

$$\xi=\xi_1=\xi_2=\frac{-8k^4m+8b_2mk^3+4(m+2)b_2^2mk^2+4k^2b_2^2(m+2)+4k^2b_2^2-(m+2)^2b_2^4}{b_2[(m+2)b_2+2k][-b_2^2(m+2)+2b_2k+4mk^2]}, \tag{C5}$$

$$k_0=\frac{4k^3(b_2+mk)}{[(m+2)b_2+2k](b_2^2(m+2)-2b_2k-4mk^2)}, \tag{C6}$$

if taking

$$\mu'=\sqrt{\xi}\mu, \quad k'=\frac{k_0}{\xi}, \tag{C7}$$

then

$$R\left\{\exp\left[-\frac{1}{k_BT}\mathcal{H}_{eff}^{\alpha}(\sigma,k)\right]\right\}=C\exp\left\{k'\mu_1'^{\alpha}\mu_2'^{\alpha}-\frac{b_{m+2}}{2}\frac{1}{m+2}(\mu_1'^{\alpha})^2-\frac{b_2}{2}\frac{1}{2}(\mu_5'^{\alpha})^2\right\}. \tag{C8}$$

The same as above for case (2), the effective Hamiltonian of the βth generator is

$$-\frac{1}{k_BT}\mathcal{H}_{eff}^{\beta}(\sigma,k)=k\left[\sigma_1^\beta\sigma_3^\beta+\sigma_4^\beta\sigma_2^\beta+\sum_{i=5}^{m+4}(\sigma_3^\beta+\sigma_3^\beta)\sigma_i^\beta\right]-\frac{b_{m+2}}{2}\left[\frac{1}{m+2}(\sigma_1^\beta)^2+(\sigma_3^\beta)^2+(\sigma_4^\beta)^2+\frac{1}{m+2}(\sigma_2^\beta)^2\right]$$
$$-\frac{b_2}{2}\left[\sum_{i=5}^{m+4}(\sigma_i^\beta)^2\right], \tag{C9}$$

where the coefficients $1/(m+2)$ in the terms $(\sigma_1^\beta)^2$ and $(\sigma_2^\beta)^2$ come from the fact that $m+2$ neighboring generators share the same sites 1 and 5. By integrating spins σ_3, σ_4, and σ_i $(i=5,\ldots,m+4)$ from $-\infty$ to $+\infty$ and using the relation (C4), the same result can be obtained,

$$R\left\{\exp\left[-\frac{1}{k_BT}\mathcal{H}_{eff}^{\beta}(\sigma^\beta,k)\right]\right\}=\int_{-\infty}^{\infty}\prod_{i=1}^{m+4}d\sigma_i^\beta T^\beta(\mu,\sigma^\beta)\exp\left[-\frac{1}{k_BT}\mathcal{H}_{eff}^{\beta}(\sigma^\beta,k)\right]$$
$$=C\exp\left\{k'\mu_1'^{\beta}\mu_2'^{\beta}-\frac{b_{m+2}}{2}\left[\frac{1}{m+2}(\mu_1'^{\beta})^2+\frac{1}{m+2}(\mu_2'^{\beta})^2\right]\right\}. \tag{C10}$$

Therefore, from (C8) and (C10), we have

$$R\{P_{eq}(k,\{\sigma\})\}=\frac{1}{Z'}\exp\left[k'\sum_{\langle i,j\rangle}\mu_i'\mu_j'-\frac{b_{q_i}}{2}\sum_i\mu_i'^2\right]=P_{eq}'(k',\{\mu'\}), \tag{C11}$$

where the recursion relation is

$$k' = \frac{4k^2(b_2+mk)b_2}{8k^4m - 8b_2mk^3 - 4(m^2+3m+3)b_2^2k^2 + (m+2)^2b_2^4}k. \tag{C12}$$

Expression (C11) means that the distribution function is invariant under RG transformation.

2. RG transformation of the master equation

Besides $R\{P_{eq}(k,\{\sigma\})\}$, one can also calculate

$$R\{\sigma_j^\alpha P_{eq}(k,\{\sigma\})\} = \sum_{\{\sigma\}} \{T(\mu,\sigma)\sigma_j^\alpha P_{eq}(k,\{\sigma\})\}$$
$$= \int_{-\infty}^{\infty} d\sigma_1\, d\sigma_2 \,\ldots\, d\sigma_N \prod_\beta \delta(\mu_1^\alpha - \sigma_1^\alpha)\delta(\mu_2^\alpha - \sigma_2^\alpha)\sigma_j^\alpha P_{eq}(k,\{\sigma\})$$
$$= P'_{eq}(k',\{\mu'\})\frac{W_{\sigma_j^\alpha}}{W},$$

where

$$W = \int_{-\infty}^{\infty} d\sigma_3^\alpha d\sigma_4^\alpha \left(\prod_{i=5}^{m+4} d\sigma_i^\alpha\right) \exp\left\{ k\left[\mu_1^\alpha\sigma_3^\alpha + \sigma_3^\alpha\sigma_4^\alpha + \sigma_4^\alpha\mu_2^\alpha + \sum_{i=5}^{m+4} (\sigma_4^\alpha + \sigma_3^\alpha)\sigma_i^\alpha \right] - \frac{b_{m+1}}{2}[(\sigma_3^\alpha)^2 + (\sigma_4^\alpha)^2] - \frac{b_2}{2}\left[\sum_{i=5}^{m+4} (\sigma_i^\alpha)^2 \right]\right\}$$
$$= \left(\frac{2\pi}{b_2}\right)^{m/2} \sqrt{\frac{(2\pi)^2 b_2}{(b_{m+2}+k)(b_2b_{m+2} - b_2k - 2mk^2)}} \exp\left\{ \frac{k^3(b_2+mk)}{(b_{m+2}+k)(b_2b_{m+2} - b_2k - 2mk^2)}\mu_1^\alpha\mu_2^\alpha + \frac{1}{2}\frac{k^2(b_2b_{m+2} - mk^2)}{(b_{m+2}+k)(b_2b_{m+2} - b_2k - 2mk^2)}[(\mu_1^\alpha)^2 + (\mu_2^\alpha)^2]\right\},$$

$$W_{\sigma_j^\alpha} = \int_{-\infty}^{\infty} d\sigma_3^\alpha d\sigma_4^\alpha \left(\prod_{i=5}^{m+4} d\sigma_i^\alpha\right) \sigma_j^\alpha \exp\left\{ k\left[\mu_1^\alpha\sigma_3^\alpha + \sigma_3^\alpha\sigma_4^\alpha + \sigma_4^\alpha\mu_2^\alpha + \sum_{i=5}^{m+4} (\sigma_4^\alpha + \sigma_3^\alpha)\sigma_i^\alpha \right] - \frac{b_{m+2}}{2}[(\sigma_3^\alpha)^2 + (\sigma_4^\alpha)^2] - \frac{b_2}{2}\left[\sum_{i=5}^{m+4} (\sigma_i^\alpha)^2 \right]\right\}$$
$$= \int_{-\infty}^{\infty} d\sigma_3^\alpha d\sigma_4^\alpha \left\{ \prod_{i\neq j=5}^{m+4} \left(\int_{-\infty}^{\infty} d\sigma_i^\alpha \exp\left[k(\sigma_4^\alpha + \sigma_3^\alpha)\sigma_i^\alpha - \frac{b_2}{2}(\sigma_i^\alpha)^2 \right]\right)\right\}$$
$$\times \left(\int_{-\infty}^{\infty} d\sigma_j^\alpha \sigma_j^\alpha \exp\left[k(\sigma_4^\alpha + \sigma_3^\alpha)\sigma_j^\alpha - \frac{b_2}{2}(\sigma_j^\alpha)^2 \right]\right) \exp\left\{ k[\mu_1^\alpha\sigma_3^\alpha + \sigma_3^\alpha\sigma_4^\alpha + \sigma_4^\alpha\mu_2^\alpha] - \frac{b_{m+2}}{2}[(\sigma_3^\alpha)^2 + (\sigma_4^\alpha)^2]\right\}$$
$$= \frac{k^2}{b_2b_{m+2} - b_2k - 2mk^2} W, \quad j = 5, \ldots, m+4,$$

$$W_{\sigma_4^\alpha}=\int_{-\infty}^{\infty} d\sigma_3^\alpha d\sigma_4^\alpha \left(\prod_{i=5}^{m+4} d\sigma_i^\alpha\right)\sigma_4^\alpha \exp\left\{k\left[\mu_1^\alpha\sigma_3^\alpha+\sigma_3^\alpha\sigma_4^\alpha+\sigma_4^\alpha\mu_2^\alpha+\sum_{i=5}^{m+4}(\sigma_4^\alpha+\sigma_3^\alpha)\sigma_i^\alpha\right]-\frac{b_{m+2}}{2}[(\sigma_3^\alpha)^2+(\sigma_4^\alpha)^2]\right.$$
$$\left.-\frac{b_2}{2}\left[\sum_{i=5}^{m+4}(\sigma_i^\alpha)^2\right]\right\}$$
$$=\frac{(mk^3+b_2k^2)\mu_1^\alpha+(b_2b_{m+2}-mk^2)k\mu_2^\alpha}{(k+b_{m+2})(b_2b_{m+2}-b_2k-2mk^2)}W,$$

$$W_{\sigma_3^\alpha}=\int_{-\infty}^{\infty} d\sigma_3^\alpha d\sigma_4^\alpha \left(\prod_{i=5}^{m+4} d\sigma_i^\alpha\right)\sigma_3^\alpha \exp\left\{k\left[\mu_1^\alpha\sigma_3^\alpha+\sigma_3^\alpha\sigma_4^\alpha+\sigma_4^\alpha\mu_2^\alpha+\sum_{i=5}^{m+4}(\sigma_4^\alpha+\sigma_3^\alpha)\sigma_i^\alpha\right]-\frac{b_{m+2}}{2}[(\sigma_3^\alpha)^2+(\sigma_4^\alpha)^2]\right.$$
$$\left.-\frac{b_2}{2}\left[\sum_{i=5}^{m+4}(\sigma_i^\alpha)^2\right]\right\}$$
$$=\frac{[(mk^2+b_2k)\mu_2^\alpha+(b_2b_{m+2}-mk^2)\mu_1^\alpha]k}{(k+b_{m+2})(b_2b_{m+2}-b_2k-2mk^2)}W,$$

and then

$$R\{\sigma_1^\alpha P_{eq}(k,\{\sigma\})\}=\frac{1}{\sqrt{\xi}}\mu_1'^\alpha P'_{eq}(k',\{\mu'\}), \tag{C13}$$

$$R\{\sigma_2^\alpha P_{eq}(k,\{\sigma\})\}=\frac{1}{\sqrt{\xi}}\mu_2'^\alpha P'_{eq}(k',\{\mu'\}), \tag{C14}$$

$$R\{\sigma_3^\alpha P_{eq}(k,\{\sigma\})\}=\frac{1}{\sqrt{\xi}}\frac{(mk^3+b_2k^2)\mu_2'^\alpha+k(b_2b_{m+2}-mk^2)\mu_1'^\alpha}{(k+b_{m+2})(b_2b_{m+2}-b_2k-2mk^2)}P'_{eq}(k',\{\mu'\}), \tag{C15}$$

$$R\{\sigma_4^\alpha P_{eq}(k,\{\sigma\})\}=\frac{1}{\sqrt{\xi}}\frac{(mk^3+b_2k^2)\mu_1'^\alpha+(b_2b_{m+2}-mk^2)k\mu_2'^\alpha}{(k+b_{m+2})(b_2b_{m+2}-b_2k-2mk^2)}P'_{eq}(k',\{\mu'\}), \tag{C16}$$

$$R\{\sigma_j^\alpha P_{eq}(k,\{\sigma\})\}=\frac{1}{\sqrt{\xi}}\frac{k^2}{b_2b_{m+2}-b_2k-2mk^2}(\mu_1'^\alpha+\mu_2'^\alpha)P'_{eq}(k',\{\mu'\}),\quad j=5,\ldots,m+4. \tag{C17}$$

By virtue of these integral results, Eqs. (3.36) and (3.37) can be obtained.

[1] B. I. Halperin and P. C. Hohenberg, Phys. Rev. Lett. **19**, 700 (1967).
[2] P. C. Hohenberg and B. I. Halperin, Rev. Mod. Phys. **49**, 435 (1977).
[3] R. J. Glauber, J. Math. Phys. **4**, 294 (1963).
[4] Jian-Yang Zhu and Z. R. Yang, Phys. Rev. E **59**, 1551 (1999).
[5] Jian-Yang Zhu and Z. R. Yang, Phys. Rev. E **61**, 210 (2000).
[6] K. G. Wilson, Phys. Rev. B **4**, 3174 (1971); **4**, 3184 (1971).
[7] S. K. Ma, *Modern Theory of Critical Phenomena* (Benjamin, New York, 1976).
[8] Y. Achiam and J. M. Kosterlitz, Phys. Rev. Lett. **41**, 128 (1978).
[9] Y. Achiam, J. Phys. A **11**, 975 (1978); **11**, L129 (1978); **13**, 1825 (1980).
[10] Y. Achiam, Physica A **120**, 279 (1983).
[11] Y. Achiam, Phys. Rev. B **19**, 376 (1979); **28**, 428 (1983); **31**, 4732 (1985); **33**, 7762 (1986).
[12] Y. Achiam, Phys. Rev. B **32**, 1796 (1985).
[13] D. Kutasov, A. Aharony, E. Domany, and W. Kinzel, Phys. Rev. Lett. **56**, 2229 (1986).
[14] P. O. Weir and J. M. Kosterlitz, Phys. Rev. B **33**, 391 (1986).
[15] E. J. Lage, J. Phys. C **19**, 5715 (1986); **19**, L91 (1986); Physica (Utrecht) **140A**, 629 (1987).
[16] G. F. Mazenko, M. J. Nolan, and O. T. Valls, Phys. Rev. Lett. **41**, 500 (1978).
[17] W. Kinzel, Z. Phys. B: Condens. Matter **29**, 361 (1978).
[18] D. Kandel, Phys. Rev. B **38**, 486 (1988).
[19] B. B. Mandelbrot, *Fractals: Form, Chance, and Dimension* (Freeman, San Francisco, 1977); *The Fractal Geometry of Nature* (Freeman, San Francisco, 1982).
[20] Y. Gefen, B. B. Mandelbrot, and A. Aharony, Phys. Rev. Lett. **45**, 855 (1980).
[21] Xiang-Mu Kong and Song Li, Sci. China, Ser. A: Math., Phys., Astron. Technol. Sci. **42**, 325 (1999).